

Quantum Modules

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Abstract

There are various generalizations of bialgebras to their “many object” versions, such as quantum categories, bialgebroids and weak bialgebras. These can also be thought of as quantum analogues of small categories. In this paper we study modules over these structures, which are quantum analogues of profunctors (also called distributors) between small categories.

1 Introduction

Notions of \times_A -coalgebra and \times_A -bialgebra were introduced by Takeuchi [9]. Takeuchi’s \times_A -bialgebras generalize bialgebras and are a special case of quantum categories [3], which are defined for an arbitrary braided monoidal category \mathcal{V} and also include small categories.

In this paper we define modules over quantum categories. Modules over \times_A -bialgebras have been considered before. However our definition is the natural one from the point of view of category theory. In the $\mathcal{V} = \text{Set}$ case it gives profunctors between small categories. Further, we discuss the question of composing such modules, analogously to composing profunctors.

First we work in an arbitrary braided monoidal category \mathcal{V} . Then we consider several special cases. In Section 3 we briefly examine the $\mathcal{V} = \text{Set}$ case. The setting of Section 4 is that of Takeuchi [9]. Here we also obtain a result about associativity of the operation \times_A . Section 5 is dedicated to weak bialgebras. Takeuchi’s operation \times_A is computed for weak bialgebras.

2 Comonads, monoidales and Kan extensions

In this section we will work with a monoidal bicategory \mathcal{B} . We assume that for every $n > 2$ a choice of an n -ary tensor product pseudofunctor

$$\mathcal{B}^n \xrightarrow{\otimes_n} \mathcal{B}$$

is made, which involves choosing an order of bracketing for the tensor product. The expression $B_1 \otimes \dots \otimes B_n$ refers to $\otimes_n(B_1 \otimes \dots \otimes B_n)$.

A comonad in \mathcal{B} is a pair (B, g) where B is an object of \mathcal{B} and $g = (g, \delta : g \Rightarrow gg, \epsilon : g \Rightarrow 1_g)$ is a comonoid in the homcategory $\mathcal{B}(B, B)$. A map $(k, \kappa) : (B, g) \longrightarrow (B', g')$ of comonads consists of a 1-cell $k : B \Rightarrow B'$ and a 2-cell $\kappa : kg \Rightarrow g'k$ satisfying the conditions

$$(kg \xrightarrow{k\delta} kgg \xrightarrow{\kappa g} g'kg \xrightarrow{g'\kappa} g'g'k) = (kg \xrightarrow{\kappa} g'k \xrightarrow{g'\delta} g'g'k),$$

$$(kg \xrightarrow{k\epsilon} k) = (kg \xrightarrow{\kappa} g'k \xrightarrow{\epsilon k} k).$$

A comonad map transformation $\tau : (k, \kappa) \Rightarrow (k', \kappa') : (A, g) \longrightarrow (B, g')$ is a 2-cell $\tau : k \Rightarrow k'$ satisfying

$$(kg \xrightarrow{\tau g} k'g \xrightarrow{\kappa'} g'k') = (kg \xrightarrow{\kappa} g'k \xrightarrow{g'\tau} g'k').$$

Comonads in \mathcal{B} , comonad maps and comonad map transformations form a bicategory $\text{Comnd}\mathcal{B}$ under the obvious compositions.

A monoidale (called “pseudomonoid” in [2]) in \mathcal{B} consists of an object E , morphisms $p : E \otimes E \rightarrow E$ and $j : I \rightarrow E$ called the multiplication and the unit respectively, and invertible 2-cells $p(p \otimes 1_E) \Rightarrow p(1_E \otimes p)$, $p(j \otimes 1_E) \Rightarrow 1_E$ and $p(1_E \otimes j) \Rightarrow 1_E$ satisfying two axioms. A monoidal morphism between monoidales $(f, \phi_2, \phi_0) : E \rightarrow D$ consists of a morphism $f : E \rightarrow D$ and 2-cells $\phi_2 : p(f \otimes f) \Rightarrow fp$, $\phi_0 : j \Rightarrow fj$ satisfying three axioms. Monoidales in \mathcal{B} , monoidal morphisms and the obvious 2-cells form a bicategory $\text{Mon}\mathcal{B}$.

For any monoidal E there is an n -ary multiplication map

$$E^n \xrightarrow{p_n} E.$$

It is defined by consecutive multiplications following the order of the chosen bracketing for the tensor product in \mathcal{B} .

A monoidal comonad is a comonad in $\text{Mon}\mathcal{B}$. Explicitly, it consists of a monoidale E a comonad g on E such that the comultiplication $\delta : g \Rightarrow gg$ and the counit $\epsilon : g \Rightarrow 1_g$ are maps of monoidal morphisms.

Suppose that E is a monoidale and $g : E \longrightarrow E$ is an endomorphism such that the left Kan extensions $\text{Lan}_p(p(g \otimes g))$, $\text{Lan}_p(p(\text{Lan}_p(p(g \otimes g)) \otimes g))$, $\text{Lan}_p(p(g \otimes \text{Lan}_p(p(g \otimes g))))$ and $\text{Lan}_{p_3}(p_3(g \otimes g \otimes g))$ exist. Giving a monoidal structure on g is equivalent to giving 2-cells $\mu : \text{Lan}_p(p(g \otimes g)) \Rightarrow g$ and $\eta : \text{Lan}_j j \Rightarrow g$ satisfying the conditions

$$\begin{array}{ccc}
& \text{Lan}_p(p(\text{Lan}_p(p(g \otimes g)) \otimes g)) & \xrightarrow{\text{Lan}_p(p(\mu \otimes g))} \text{Lan}_p(p(g \otimes g)) \\
& \nearrow & \searrow \mu \\
\text{Lan}_{p_3}(p_3(g \otimes g \otimes g)) & & \\
& \searrow & \nearrow \mu \\
& \text{Lan}_p(p(g \otimes \text{Lan}_p(p(g \otimes g)))) & \xrightarrow{\text{Lan}_p(p(g \otimes \mu))} \text{Lan}_p(p(g \otimes g))
\end{array}$$

$$\begin{array}{ccc}
g & \xrightarrow{1_g} & g \\
\downarrow & & \uparrow \mu \\
\text{Lan}_p(p(\text{Lan}_j j \otimes g)) & \xrightarrow{\text{Lan}_p(p(\eta \otimes g))} & \text{Lan}_p(p(g \otimes g))
\end{array} \tag{1}$$

$$\begin{array}{ccc}
g & \xrightarrow{1_g} & g \\
\downarrow & & \uparrow \mu \\
\text{Lan}_p(p(g \otimes \text{Lan}_j j)) & \xrightarrow{\text{Lan}_p(p(g \otimes \eta))} & \text{Lan}_p(p(g \otimes g))
\end{array}$$

The unnamed arrows here and below are the canonical maps, determined by the universal properties of left Kan extensions.

Suppose that (B, g) is a comonad and $k : B \rightarrow B'$ is a 1-cell. Assume that the left Kan extension $\text{Lan}_k(kg)$ exists and let $\kappa : kg \Rightarrow \text{Lan}_k(kg)k$ be the universal 2-cell. The pair $(B', \text{Lan}_k kg)$ can be uniquely turned into a comonad so that (k, κ) becomes a comonad map [6]. Furthermore, there is a correspondence between comonad maps:

$$\frac{(B, g) \xrightarrow{(k, \kappa)} (B', g')}{(B', \text{Lan}_k g) \xrightarrow{(1_{B'}, \kappa')} (B', g')}$$

Or more precisely there is an equivalence of categories:

$$\text{Comnd}\mathcal{B}((B, g), (B', g')) \simeq \text{Comon}\mathcal{B}(\text{Lan}_k kg, B'). \tag{2}$$

Comon stands for the category of comonoids.

Suppose that E is a monoidale and g is a comonad on E . Using (2) it can be seen that giving a monoidal structure on the comonad g is equivalent to giving comonoid maps

$\mu : Lan_p(g \otimes g) \longrightarrow g$ and $\eta : Lan_j j \longrightarrow g$ such that the diagrams (1) commute, now in the category $Comon\mathcal{B}(E, E)$.

The reader might recognize the appropriateness of the context of multitensor categories. Provided certain left Kan extensions exist, a monoidal structure on E determines a lax monoidal structure on $\mathcal{B}(E, E)$. The n -ary tensor product is

$$Lan_{p_n}(p_n(- \otimes \dots \otimes -)).$$

Notion of the comonoid makes sense in any multitensor category. A monoid in $\mathcal{B}(E, E)$ is a monoidal endomorphism on E . The multitensor structure of $\mathcal{B}(E, E)$ can be lifted to $Comon\mathcal{B}(E, E)$. A monoid in $Comon\mathcal{B}(E, E)$ is a monoidal comonad on E .

Definition 1. For monoidales E and E' , an (E, E') -actee is a pseudoalgebra for the pseudomonad $E \otimes - \otimes E'$ on \mathcal{B} . A map between actees is a map of pseudoalgebras.

An (E, E') -actee structure on an object B consists of a morphism $a : E \otimes B \otimes E' \longrightarrow B$ and isomorphisms $a(1_E \otimes a \otimes 1_{E'}) \Rightarrow a(p \otimes 1_B \otimes p)$, $a(j \otimes 1_B \otimes j) \Rightarrow 1_B$ satisfying the two axioms. Here are two special cases of this concept:

Definition 2. Suppose that $g : E \longrightarrow E$ and $g' : E' \longrightarrow E'$ are monoidal endomorphisms. A (g, g') -action on an endomorphism $m : B \longrightarrow B$ consists of a morphism $a : E \otimes B \otimes E' \longrightarrow B$ and a 2-cell $\gamma : a(g \otimes m \otimes g') \Rightarrow ma$ satisfying axioms.

Definition 3. Suppose that (E, g) and (E', g') are monoidal comonads. A (g, g') -action on a comonad (B, m) consists of a morphism $a : E \otimes B \otimes E' \longrightarrow B$ and a comonad map of the form $(a, \gamma) : (E \otimes B \otimes E', g \otimes m \otimes g') \longrightarrow (B, m)$ satisfying axioms.

In both cases there is an underlying (E, E') -action on the object B . With existence of the left Kan extensions, a (g, g') -action on m , with a given underlying (E, E') -action on B , is determined by a 2-cell $\alpha : Lan_a(a(g \otimes m \otimes g')) \Rightarrow m$ satisfying two axioms:

$$\begin{array}{ccc}
Lan_a(a(g \otimes Lan_a(a(g \otimes m \otimes g')) \otimes g')) & \xrightarrow{Lan_a(a(g \otimes \alpha \otimes g'))} & Lan_a(a(g \otimes m \otimes g')) \\
\uparrow \swarrow & & \searrow \alpha \\
Lan_{a_3}(a_3(g \otimes g \otimes m \otimes g' \otimes g')) & & m \\
\downarrow \searrow & & \nearrow \alpha \\
Lan_a(a(Lan_p(p(g \otimes g)) \otimes m \otimes Lan_p(p(g' \otimes g')))) & \xrightarrow{Lan_a(a(\mu \otimes m \otimes \mu))} & Lan_a(a(g \otimes m \otimes g'))
\end{array}$$

$$\begin{array}{ccc}
m & \xrightarrow{1_m} & m \\
\downarrow & & \uparrow \alpha \\
Lan_a(a(Lan_j j \otimes m \otimes Lan_j j)) & \xrightarrow{Lan_a(a(\eta \otimes m \otimes \eta))} & Lan_a(a(g \otimes m \otimes g'))
\end{array}
\tag{3}$$

In the case of comonads, α should be a comonoid map, and the diagrams above should commute in $\text{Comon}\mathcal{B}(B, B)$.

For a (g, g') -action on m , the left action map α_l is defined to be the composite:

$$\text{Lan}_{a_l}(a_l(g \otimes m)) \rightarrow \text{Lan}_a(a(g \otimes m \otimes \text{Lan}_j j)) \xrightarrow{\text{Lan}_a(a(g \otimes m \otimes \eta))} \text{Lan}_a(a(g \otimes m \otimes g')) \xrightarrow{\alpha} m$$

and the right action map α_r is defined to be the composite:

$$\text{Lan}_{a_r}(a_r(m \otimes g')) \rightarrow \text{Lan}_a(a(\text{Lan}_j j \otimes m \otimes g')) \xrightarrow{\text{Lan}_a(a(g \otimes m \otimes \eta))} \text{Lan}_a(a(g \otimes m \otimes g')) \xrightarrow{\alpha} m,$$

where $a_l = a(1_E \otimes j)$ and $a_r = (j \otimes 1'_E)$.

3 Quantum Modules

Let $\mathcal{V} = (\mathcal{V}, \otimes, c)$ be a braided monoidal category. Assume that each of the functors $X \otimes -$ preserves coreflexive equalizers.

We will work with a monoidal bicategory $\text{Comod}\mathcal{V}$ considered in [2], which will be taken as the monoidal bicategory \mathcal{B} of the previous section. Objects of $\text{Comod}\mathcal{V}$ are the comonoids $C = (C, \delta : C \rightarrow C \otimes C, \epsilon : C \rightarrow I)$ in \mathcal{V} . The homcategory $\text{Comod}\mathcal{V}(C, D)$ is the category of Eilenberg-Moore coalgebras for the comonad $C \otimes - \otimes D$. A 1-cell from C to D , depicted as $C \dashrightarrow D$, is a comodule from C to D . The composition $N \circ M$ is defined by a coreflexive equalizer

$$M \otimes_C N \rightarrow M \otimes N \xrightleftharpoons[1 \otimes \delta_l]{\delta_r \otimes 1} M \otimes C \otimes N.$$

The monoidal structure of $\text{Comod}\mathcal{V}$ extends that of \mathcal{V} (although it is not braided). Each comonoid $C = (C, \delta, \epsilon)$ has an opposite comonoid $C^o = (C, c\delta, \epsilon)$. There are comodules

$$e : C \otimes C^o \dashrightarrow I \quad n : I \dashrightarrow C^o \otimes C,$$

both of which are C as objects with coactions in string notation respectively:



This exhibits C^o as a right bidual of C in $\text{Comod}\mathcal{V}$. It follows that $C^o \otimes C$ is a monoidale with multiplication $p = C^o \otimes e \otimes C$ and unit $j = n$.

Throughout this paper a right $C^o \otimes C'$ -comodule X will be regarded as a comodule $C \dashrightarrow C'$ using biduality when tensor products $X \otimes_{C'} -$ or $- \otimes_C X$ are taken.

Let $X_1 : I \dashrightarrow A \otimes C^o$ and $X_2 : I \dashrightarrow C \otimes B$ be comodules. With little calculation it can be established that the composite comodule

$$I \xrightarrow{X_1 \otimes X_2} A \otimes C^o \otimes C \otimes B \xrightarrow{A \otimes e \otimes B} A \otimes B$$

is $X_1 \otimes X_2$ with the right A - and B -coactions on it induced from the right A -coaction on X_1 and the right B -coaction on X_2 .

Definition 4. An algebroid in \mathcal{V} is a pair (A, C) , where C is a comonoid in \mathcal{V} and A is a monoidal endomorphism on $C^o \otimes C$.

Definition 5. (see [3]) A quantum category in \mathcal{V} is a pair (A, C) , where C is a comonoid in \mathcal{V} and A is a monoidal comonad on $C^o \otimes C$.

For comonoids C and C' , the map $a = C \otimes e \otimes e \otimes C'$ determines a $(C^o \otimes C, C'^o \otimes C')$ -action on $C^o \otimes C'$.

Definition 6. A module from an algebroid (A, C) to an algebroid (A', C') consists of a comodule $M : C'^o \otimes C \dashrightarrow C'^o \otimes C$ and a (A, A') -action on M , such that the underlying $(C^o \otimes C, C'^o \otimes C')$ -action on $C^o \otimes C$ is the canonical one.

Definition 7. A quantum module from a quantum category (A, C) to a quantum category (A', C') consists of a comonad $(M, C'^o \otimes C)$ and an (A, A') -action on $(M, C'^o \otimes C)$, such that the underlying $(C^o \otimes C, C'^o \otimes C')$ action on $C^o \otimes C'$ is the canonical one.

A (quantum) module M from (A, C) to (A', C') has a coaction 2-cell:

$$\begin{array}{ccc} C^o \otimes C \otimes C^o \otimes C' \otimes C'^o \otimes C' & \xrightarrow{A \otimes M \otimes A'} & C^o \otimes C \otimes C^o \otimes C' \otimes C'^o \otimes C' \\ \downarrow C^o \otimes e \otimes e \otimes C' & \gamma \Downarrow & \downarrow C^o \otimes e \otimes e \otimes C' \\ C^o \otimes C' & \xrightarrow{M} & C^o \otimes C' \end{array}$$

satisfying two axioms. In the case of a quantum module $(C^o \otimes e \otimes e \otimes C', \gamma)$ should be a comonad map.

A map of (quantum) modules is a comodule map $M_1 \rightarrow M_2$ respecting the action (for quantum modules it also should be a comonad transformation).

We will apply the machinery of Section 2 to our present context. For this we will need existence of certain left Kan extensions in $\text{Comod}\mathcal{V}$, and that will be discussed prior. First we introduce the following structure on the class of comodules of the form $X : C^o \otimes C' \dashrightarrow C^o \otimes C'$ (strictly speaking on the class of triples (X, C, C') , where C and C' are comonoids and X is a comodule of the indicated form).

For comodules $X_i : C_{i-1}^o \otimes C_i \rightharpoonup C_{i-1}^o \otimes C_i$, $1 \leq i \leq n$, define $T_{(C_0, C_2, \dots, C_n)}(X_1, X_2 \dots X_n)$ or simply $T_n(X_1, X_2 \dots X_n)$ to be the comodule determined by the left Kan extension

$$\begin{array}{ccc}
C_0^o \otimes C_1 \otimes \dots \otimes C_{n-1} \otimes C_n & \xrightarrow{X_1 \otimes \dots \otimes X_n} & C_0^o \otimes C_1 \otimes \dots \otimes C_{n-1} \otimes C_n \\
\downarrow C^o \otimes e \otimes \dots \otimes e \otimes C_n & \Downarrow & \downarrow C^o \otimes e \otimes \dots \otimes e \otimes C_n \\
C_0^o \otimes C_n & \xrightarrow{T_n(X_1, \dots, X_n)} & C_0^o \otimes C_n
\end{array} \quad (4)$$

when this exists. For $n = 1$ this gives $T_1(X_1) = X_1$. For a comonoid C define $T_C()$ or simply $T_0()$ to be the comodule determined by the left Kan extension

$$\begin{array}{ccc}
& I & \\
n \swarrow & \Downarrow & \searrow n \\
C^o \otimes C & \xrightarrow{T_0()} & C^o \otimes C
\end{array}$$

when this exists. Clearly, the T_n can be made into functors.

For each partition $\xi : m = m_1 + m_2 + \dots + m_n$, $m_i \geq 0$, the universal properties of left Kan extensions give an associativity map:

$$\beta_\xi : T_m(X_{11}, \dots, X_{nm_m}) \longrightarrow T_n(T_{m_1}(X_{11}, \dots, X_{1m_1}), \dots, T_{m_n}(X_{n1}, \dots, X_{nm_n})). \quad (5)$$

These are natural in all variables and satisfy coherence conditions.

When it exists, let $\text{coHom}(X, Y)$ be the internal cohom object in \mathcal{V} , meaning that there is a natural bijection:

$$\frac{Y \xrightarrow{f} X \otimes Z}{\text{coHom}(X, Y) \xrightarrow{f^*} Z}$$

If X and Y are left C -comodules, then $\text{coHom}_C(X, Y)$ is defined by the coequalizer

$$\text{coHom}(C \otimes X, Y) \xrightleftharpoons[(\delta_l^Y \text{coev}_{X,Y})^*]{(\text{coev}_{X,Y} \delta_l^X)^*} \text{coHom}(X, Y) \longrightarrow \text{coHom}_C(X, Y).$$

If $X : C \rightharpoonup A$ and $Y : C \rightharpoonup B$ are comodules, then $\text{coHom}_C(X, Y)$ becomes a $A \rightharpoonup B$ comodule. The left Kan extension of Y along X is $\text{coHom}(X, Y)$.

We deduce that the left Kan extensions (4) exist if $\text{coHom}(C, X)$ exists for every X . For $n = 2$, $T_2(X_1, X_2)$ can be computed as (setting for simplicity of notation $C_1 = C$):

$$\begin{aligned} \text{coHom}_{(C^o \otimes C \otimes C^o \otimes C)}(C \otimes C \otimes C, (C_0 \otimes e \otimes C_2) \circ (X_1 \otimes X_2)) &\cong \\ \text{coHom}_{(C^o \otimes C \otimes C^o \otimes C)}(C \otimes C \otimes C, X_1 \otimes_C X_2) &\cong \\ \text{coHom}_{(C \otimes C^o)}(C, X_1 \otimes_C X_2). \end{aligned} \quad (6)$$

It can be shown that this is isomorphic to the coequalizer of the pair

$$\text{coHom}(C, X_1 \otimes_C X_2) \underset{(\delta_l^{C^o})^*}{\overset{(\delta_l^C)^*}{\rightrightarrows}} X_1 \otimes_C X_2, \quad (7)$$

wherein the left C -coaction δ_l^C on $X_1 \otimes X_2$ is induced by the left coaction of C on X_1 , and the left C^o -coaction $\delta_l^{C^o}$ on $X_1 \otimes X_2$ is induced by the left coaction of C^o on X_2 .

For $n = 0$ we have

$$T_C() = \text{coHom}(C, C).$$

The next lemma provides an even more general situation when the operations T_n can be defined. We need the following definition.

A $C \otimes$ -coequalizer of the pair of morphisms in \mathcal{V}

$$Y \underset{g}{\overset{f}{\rightrightarrows}} C \otimes X$$

is a map $h : X \rightarrow Z$ for which $(1 \otimes h)f = (1 \otimes h)g$ such that for any other map $h' : X \rightarrow Z'$ for which $(1 \otimes h')f = (1 \otimes h')g$ there exists a unique $z : Z \rightarrow Z'$ with $zh = h'$.

Lemma 8. *Suppose that Y is a $A \otimes C \otimes C^o \otimes B \rightarrowtail D$ comodule. Suppose that Z is the $C \otimes$ -coequalizer of the pair*

$$Y \underset{\delta_l^{C^o}}{\overset{\delta_l^C}{\rightrightarrows}} C \otimes Y. \quad (8)$$

Then Z becomes a comodule $A \otimes B \rightarrowtail D$. The left Kan extension of Y along $A \otimes e \otimes B$ is Z .

$$\begin{array}{ccc}
& A \otimes C^o \otimes C \otimes B & \\
A \otimes e \otimes B \swarrow & \Downarrow & \searrow Y \\
A \otimes B & \xrightarrow{Z} & D
\end{array}$$

Proof. Let $h : Y \rightarrow Z$ be the $C \otimes$ -coequalizer of (8). In the diagram

$$\begin{array}{ccccc}
Y & \xrightarrow[\delta_l^{C^o}]{\delta_l^C} & C \otimes Y & \xrightarrow{1 \otimes h} & C \otimes Z \\
\delta_l^A \downarrow & & \downarrow c^{-1} \delta_l^A & & \downarrow 1 \otimes \delta_l^A \\
A \otimes Y & \xrightarrow[A \otimes \delta_l^{C^o}]{A \otimes \delta_l^C} & A \otimes C \otimes Y & \xrightarrow{(c \otimes 1)(1 \otimes 1 \otimes h)} & C \otimes A \otimes Z
\end{array}$$

the top and the bottom horizontal parts are commutative and so are the two squares to the left. Hence there exists a unique map $\delta_l^A : Z \rightarrow A \otimes Z$ rendering the right square commutative. This map defines left A -coaction on Z . Left B - and right D -coactions on Z are defined similarly. Thus Z becomes a comodule $A \otimes B \rightarrow D$.

For a comodule $X : A \otimes B \rightarrow D$ the composite comodule $X \circ (1 \otimes e \otimes 1)$ is $C \otimes X$. The left A - and B - and right D -coactions on $C \otimes X$ are induced by the respective coactions on X . The left C - and C^o -coactions on X are the cofree coactions, meaning that they are determined by comultiplications. Using this fact we can establish that a 2-cell

$$Y \Rightarrow X \circ (1 \otimes e \otimes 1)$$

is a map $h' : Y \rightarrow C \otimes X$ which respects left A - and B - and right D -coactions and satisfies

$$Y \xrightarrow[\delta_l^{C^o}]{\delta_l^C} C \otimes Y \xrightarrow{1 \otimes h'} C \otimes X.$$

Define the univral 2-cell

$$Y \Rightarrow Z \circ (1 \otimes e \otimes 1)$$

to be the map $(1 \otimes h)\delta_l^C : Y \rightarrow C \otimes Z$. The univral property follows from the above and the definition of $C \otimes$ -coequalizer. \square

Taking $A = C_0^o, C = C_1, B = C_2, D = C_0^o \otimes C_2$ and $X = p \circ (X_1 \otimes X_2) = X_1 \otimes_C X_2$ we get $T_2(X_1, X_2)$ to be the $C \otimes$ -coequalizer of

$$X_1 \otimes_C X_2 \xrightarrow[\delta_l^{C^o}]{\delta_l^C} C \otimes (X_1 \otimes_C X_2). \quad (9)$$

(as before we have rendered $C_1 = C$). If the internal cohom exists, then we can transpose C to the left and that will get us exactly the coequalizer diagram (7).

For $n \geq 2$, we can write $C_0^o \otimes e \otimes \dots \otimes e \otimes C_n$ as a composite

$$(C^o \otimes e \otimes C_n) \circ (C^o \otimes e \otimes C_{n-1} \otimes C_{n-1}^o \otimes C_n) \circ \dots \circ (C_0^o \otimes e \otimes C_2 \otimes \dots \otimes C_n)$$

The left Kan extensions along $C_0^o \otimes e \otimes \dots \otimes e \otimes C_n$ can be computed by consecutive applications of Lemma 8. In particular T_n for $n > 2$ can be computed in this way.

Assume henceforth that the operations T_n are defined. From Section 2 we obtain the following alternative definitions of algebroids, quantum categories and their modules:

An algebroid (A, C) in \mathcal{V} consists of a comodule $A : C^o \otimes C \dashrightarrow C^o \otimes C$ together with maps $\mu : T_2(A, A) \rightarrow A$ and $\eta : T_0() \rightarrow A$ satisfying three conditions.

A module M from an algebroid (A, C) to an algebroid (A', C') consists of a comodule $M : C^o \otimes C' \dashrightarrow C^o \otimes C'$ and a map $\alpha : T_3(A, M, A') \rightarrow M$ satisfying two conditions.

A quantum category (A, C) in \mathcal{V} consists of a comonad A on $C^o \otimes C$ together with comonoid maps $\mu : T_2(A, A) \rightarrow A$ and $\eta : T_0() \rightarrow A$ satisfying three conditions.

A quantum module M from a quantum category (A, C) to a quantum category (A', C') consists of a comonad M on $C^o \otimes C$ and a comonoid map $\alpha : T_3(A, M, A') \rightarrow M$ satisfying three conditions.

For a (quantum) module the left action map α_l and the right action map α_r are given as:

$$\alpha_l = \left(T_2(A, M) \xrightarrow{0+2} T_3(A, M, T_0()) \xrightarrow{T_3(A, M, \eta)} T_3(A, M, A') \xrightarrow{\alpha} M \right)$$

$$\text{and } \alpha_r = \left(T_2(M, A) \xrightarrow{2+2} T_3(T_0(), A, M) \xrightarrow{T_3(\eta, M, A')} T_3(A, M, A') \xrightarrow{\alpha} M \right)$$

Suppose that $(A_1, C_1), (A_2, C_2)$ and (A_3, C_3) are algebroids. Let M_1 be a module from (A_1, C_1) to (A_2, C_2) and let M_2 be a module from (A_2, C_2) to (A_3, C_3) . Define $M \bullet N$ by the coequalizer

$$T_3(M_1, A_2, M_2) \xrightleftharpoons[T_2(\alpha_r, M_2)\beta_{2+1}]{T_2(M_1, \alpha_l)\beta_{1+2}} T_2(M_1, M_2) \rightarrow M_1 \bullet M_2 \quad (10)$$

in $\text{Comod}\mathcal{V}(C_1^o \otimes C_3, C_1^o \otimes C_3)$. Coequalizers in the comodule category are computed as in \mathcal{V} .

Generally the operation \bullet is not associative (which is not surprising since T_2 itself is not associative). It is not even a proper composition since $M_1 \bullet M_2$ does not become a module from (A_1, C_1) to (A_3, C_3) . However it does have left and right units.

Given an algebroid (A, C) , via the algebroid multiplication, A becomes a module from (A, C) to (A, C) . So the action map is $\mu : T_2(A, A) \rightarrow A$.

Lemma 9. *We have: $A \bullet M = M$ and $M \bullet A = M$.*

Proof. The diagram

$$T_3(A, A, M) \xrightleftharpoons[T_2(\mu, 1)\beta_{2+1}]{T_2(1, \alpha_l)\beta_{1+2}} T_2(A, M) \xrightarrow{\alpha_l} 1$$

is a split coequalizer diagram split by the maps:

$$T_3(A, A, M) \xleftarrow{T_3(\eta, 1, 1)\beta_{0+1+1}} T_2(A, M) \quad \text{and} \quad T_2(A, M) \xleftarrow{T_2(\eta, 1)\beta_{0+1}} M.$$

This follows from the calculations below. Aside from algebroid and module axioms we use the naturality and coherence of the maps β_ξ .

$$\alpha_l T_2(\eta, 1)\beta_{0+1} = 1;$$

$$\begin{aligned} T_2(\mu, 1)\beta_{2+1}T_3(\eta, 1, 1)\beta_{0+1+1} &= T_2(\mu, 1)T_2(T_2(\eta, 1), 1)\beta_{2+1}\beta_{0+1+1} \\ &= T_2(\mu, T_2(\eta, 1), 1)T_2(\beta_{0+1}, 1) = 1; \end{aligned}$$

$$\begin{aligned} T_2(1, \alpha_l)\beta_{1+2}T_3(\eta, A, M)\beta_{0+1+1} &= T_2(1, \alpha_l)T_2(\eta, T_2(1, 1))\beta_{1+2}\beta_{0+1+1} \\ &= T_2(\eta, \alpha_l)\beta_{0+2} = T_2(\eta, 1)T_2(1, \alpha_l)\beta_{0+2}T_2(\eta, 1)\beta_{0+2}\alpha_l \\ &= T_2(\eta, 1)\beta_{0+1}\alpha_l \end{aligned}$$

At the end of the last calculation we used the fact that $\beta_{0+1} = \beta_{0+2}$. This follows directly from the definitions of β_ξ .

We have proved that $A \bullet M = M$. The proof of $A \bullet M = M$ is similar. □

To make \bullet into an associative composition we need to restrict the class of algebroids and modules that we consider.

Let \mathcal{X} be a class of comodules of the form $X : C^o \otimes C' \dashrightarrow C^o \otimes C'$ such that

1. If X_1, \dots, X_n are in \mathcal{X} , then the left Kan extension (4) exists.
2. If X_{11}, \dots, X_{mn} are in \mathcal{X} , then the map β_ξ (5) is an isomorphism for any partition $\xi = m_1 + \dots + m_n$ with $m_i > 0$.
3. If X is in \mathcal{X} , then the functors

$$X \otimes_C - : \text{Comod}\mathcal{V}(C, I) \longrightarrow \mathcal{V}$$

$$- \otimes_C X : \text{Comod}\mathcal{V}(I, C) \longrightarrow \mathcal{V}$$

preserve reflexive coequalizers.

4. If X and Y are in \mathcal{X} , then so is $X \otimes_C Y$.
5. If X is in \mathcal{X} and $X \rightarrow Y$ is an epimorphism, then Y is in \mathcal{X} .

Theorem 10. *Fix a class \mathcal{X} as above. Consider those algebroids and those modules between them for which the underlying $C'^o \otimes C \rightarrow C'^o \otimes C$ comodules are in \mathcal{X} . These form a bicategory under the composition \bullet .*

Proof. The functor $T_2(X, -)$ can be written as a composition of $X \otimes_C -$ and $\text{Lan}_p -$. $\text{Lan}_p -$ preserves coequalizers since it is a left adjoint and $X \otimes_C -$ preserves reflexive coequalizers by condition 3. We deduce that if X is in \mathcal{X} , then $T_2(X, -)$ preserves reflexive coequalizers. Similarly, $T_2(-, X)$ preserves reflexive coequalizers. So, if A_1 and A_3 are in \mathcal{X} , by the usual argument $M \bullet N$ can be made into a module from (A_1, C_1) to (A_3, C_3) . This works even when β_ξ are not isomorphisms, although this may not be evident. However given the condition of the theorem we can as well assume that β_{xi} are isomorphisms.

The role of 2-cells in our bicategory are played by module maps. The operation \bullet naturally extends to module maps giving the horizontal composition of 2-cells.

Under the condition 2, T_2 is associative up to coherent isomorphisms. Then \bullet is also associative up to coherent isomorphisms, and these isomorphisms are module maps.

The unit 1-cells are provided by Lemma 9.

To get a bicategory we only need to show that $M_1 \bullet M_2$ is in \mathcal{X} provided M_1 and M_2 are in \mathcal{X} . This is guaranteed by conditions 4 and 5 since $M_1 \bullet M_2$ is a quotient of $T_2(M_1, M_2)$ which itself is a quotient of $M_1 \otimes_C M_2$. □

The operation \bullet can be lifted to quantum modules between quantum categories by considering the coequalizer (10) in $\text{ComonComod}\mathcal{V}(C_1^o \otimes C_3, C_1^o \otimes C_3)$. Coequalizers in the latter are again computed as in \mathcal{V} . We have:

Theorem 11. *Let \mathcal{X} be as above. Consider those quantum categories and those quantum modules between them for which the underlying comodules $C^o \otimes C' \rightarrow C^o \otimes C'$ are in \mathcal{X} . These form a bicategory under the composition \bullet .*

4 The Set case

We take \mathcal{V} to be *Set* with the monoidal structure the cartesian product. Then, as pointed out for example in [3], $\text{Comod}\mathcal{V} = \text{Span}$. A comodule $X : C^o \otimes C' \rightarrow C^o \otimes C'$ is a span of the form:

$$\begin{array}{ccc}
 & X & \\
 (t', s') \swarrow & & \searrow (t, s) \\
 C \times C' & & C \times C'
 \end{array} \tag{11}$$

A comonad structure on a span like this is the property $t' = t$ and $s' = s$. The diagram (9) becomes

$$X_1 \times_C X_2 \begin{array}{c} \xrightarrow{t'_1 pr_1} \\ \xrightarrow{s'_2 pr_2} \end{array} C \times (X_1 \times_C X_2) \quad (12)$$

where $X_1 \times_C X_2$ is the pullback of

$$X_1 \xrightarrow{t_1} C \xleftarrow{s_2} X_2$$

If X_1 and X_2 are comonads, then $t'_1 pr_1 = t pr_1 = s pr_2 = s'_2 pr_2$. Thus, in this case the two parallel arrows in (12) are equal. Then the $C \times$ -coequalizer exists and is $X_1 \times_C X_2$ itself. It follows that for the comonads spans the operations T_n , $n > 2$, are defined and given by

$$T_n(X_1 \dots X_n) = X_1 \times_{C_1} \dots \times_{C_{n-1}} X_n.$$

$T_C() = C$ which is a span $C^o \otimes C \rightrightarrows C^o \otimes C$ with both legs the diagonal maps. The maps β_ξ (5) are obviously isomorphisms. The functors $- \times_C X$ and $X \times_C -$ preserve coreflexive equalizers since Set is a locally closed category. Consequently the class of comonad spans satisfies all the conditions of Theorem 10.

Quantum categories in Set are the ordinary small categories [3]. Quantum modules are the profunctors. The operation \bullet coincides with the usual composition of profunctors. The bicategory of the Theorem 10 is Prof.

The category Set can be replaced with any locally closed finitely complete category. In this case quantum categories will be the internal categories and the quantum modules will be the internal profunctors [4].

5 Comodules of bialgebroids

In this section we consider our theory for $\mathcal{V} = (k\text{-Mod})^{\text{op}}$ where k is a commutative ring. Note that in this case limits in \mathcal{V} are the colimits in $k\text{-Mod}$, the cohom objects in \mathcal{V} are the hom objects in $k\text{-Mod}$ and so on. The nomenclature is dual to that of Section 3. Nevertheless, we will freely refer to Section 3, so the reader should be somewhat careful.

The objects of $\text{Comod}(k\text{-Mod})^{\text{op}}$ are the k -algebras R , morphisms are the two sided modules between k -algebras. The category $k\text{-Mod}$ is closed, so right Kan extensions exist in $\text{Comod}(k\text{-Mod})^{\text{op}}$.

The operation T_2 is exactly the product \times_R of Takeuchi [9]. By (6) it is equal to

$$\text{Hom}_{(R^o \otimes R)}(R, X \otimes_R Y).$$

It can be also computed using (9) to yield:

$$X \times_R Y = \left\{ \sum_i m_i \otimes_R n_i \in M \otimes_R N : \sum_i (x \otimes 1) m_i \otimes_R n_i = \sum_i m_i \otimes_R (x \otimes 1) n_i \quad \forall x \in R \right\}.$$

For $n = 0$:

$$T_R() = \text{Hom}(R, R).$$

The ternary operation of Takeuchi $(- \times_R - \times_R -)$ is a special case of a slightly more general $(- \times_R - \times_S -)$, which is our T_3 . Takeuchi's maps

$$\alpha : (X \times_R Y) \times_R Z \longrightarrow X \times_R Y \times_R Z$$

$$\alpha' : X \times_R (Y \times_R Z) \longrightarrow X \times_R Y \times_R Z$$

are nothing but our β_{2+1} and β_{1+2} . Generally we set

$$T_{(R_1, \dots, R_{n-1})}(X_1 \dots X_n) = X_1 \times_{R_1} \dots \times_{R_{n-1}} X_n.$$

For a right T module we write X^T . Given a module $X : R^o \otimes R' \twoheadrightarrow R^o \otimes R'$, when tensor products $- \otimes_R X$ and $X \otimes_{R'} -$ are taken, X is regarded as a left R -module and a right R' -module by the right $R^o \otimes R'$ action, thus as X^{R^o} and $X^{R'}$. In contrast, when X appears in homs the left $R^o \otimes R'$ -action is used.

Lemma 12. *If X^T and Y^S are projective modules, then $(X \otimes_T Y)^S$ is a projective module.*

Proof. This follows from the fact that if Y^S is projective then the functor

$$(- \otimes_T Y)^S : \text{Mod}(k, T) \longrightarrow \text{Mod}(T, S)$$

preserves projective objects since it is a left adjoint to an epi-preserving functor $\text{Hom}_{S^o}(Y, -)^T$. \square

As an immediate consequence we have:

Lemma 13. *If $X_i^{R_i}$ are projective modules, then $(X_1 \otimes_{R_1} \dots \otimes_{R_{n-1}} X_n)^{R_n}$ is a projective module.*

We say that a right (left) module is a union of projectives if it is union of all of its projective submodules.

Lemma 14. *If Y^{T^o} is flat and X^T and Y^S are unions of projectives, then $(X \otimes_R Y)^S$ is a union of projectives.*

Proof. We can write $X = \text{colim} X_i$ and $Y = \text{colim} Y_j$, where X_i^T and Y_j^S are projective modules and the colimits are taken over filtered diagrams whose arrows are injections. We have

$$X \otimes_T Y = \text{colim} X_i \otimes_T Y = \text{colim} X_i \otimes_T Y_j$$

The latter colimit is over a filtered diagram whose arrows are injections again since Y^{T^o} and X_i^T are flat. Then $(X \otimes_R Y)^S$ is a union of projectives since each of $(X_i \otimes_T Y_j)^S$ is projective by the previous lemma. \square

Lemma 15. *If the $X_i^{R_i}$ are unions of projectives and the $X_i^{R_i^o}$ are flat, then the right module $(X_1 \otimes_{R_1} \dots \otimes_{R_{n-1}} X_n)^{R_n}$ is a union of projectives.*

The next two lemmas are slight modifications of Lemma 13 and Lemma 15 and their proofs are similar.

Lemma 16. *If the $X_i^{R_i^o}$ are projective modules, then $(X_1 \otimes_{R_1} \dots \otimes_{R_{n-1}} X_n)^{R_n}$ is a projective module.*

Lemma 17. *If the $X_i^{R_i}$ are unions of projectives and the $X_i^{R_i^o}$ are flat, then the right module $(X_1 \otimes_{R_1} \dots \otimes_{R_{n-1}} X_n)^{R_n^o}$ is a union of projectives.*

Recall from ring theory that a ring T is called right hereditary if any submodule of a projective right module over T is again projective. T is called hereditary if both T and T^o are right hereditary.

Lemma 18. *Every submodule of a union of projectives over a hereditary ring is a union of projectives.*

Proof. Obvious. \square

Various conditions under which β_{2+1} and β_{1+2} are bijective were given in [9] and [8]. We will obtain a similar result for an arbitrary partition $\xi = m_1 + \dots + m_n$ with $m_i > 0$.

Theorem 19. *For any partition $\xi = m_1 + \dots + m_n$ with $m_i > 0$, the map β_ξ (5) is an isomorphism, if the base rings are hereditary and each of X_{ij} is a union of projectives both as a left module and a right module.*

Proof. Suppose that S and T are rings and $A : S \twoheadrightarrow I$, $Y : S \twoheadrightarrow T$ and $Z : I \twoheadrightarrow T^o$ are modules. There is a natural map:

$$\text{Hom}_S(A, Y \otimes_T Z) \longrightarrow \text{Hom}_S(A, Y) \otimes_T Z$$

It can be easily seen that if A is a finitely generated left S module and $Z^{T^{op}}$ is projective then this map is an isomorphism. Also, if A is a finitely generated left S -module and Y^T is projective then there is an isomorphism:

$$\text{Hom}_S(A, Y \otimes_T Z) \longrightarrow Y \otimes_T \text{Hom}_S(A, Y)$$

Suppose that S^i are rings for $i = 1 \dots n$. If for each i , A_i is a finitely generated left S_i -module and $L_i^{T_i}$ and $\text{Hom}_{A_i}(A_i, L_i)^{T_{i-1}^o}$ are projective, then we have:

$$\begin{aligned} & \text{Hom}_{S_1 \otimes \dots \otimes S_n}(A_1 \otimes \dots \otimes A_n, L_1 \otimes_{T_1} \dots \otimes_{T_{n-1}} L_n) \cong \\ & \text{Hom}_{S_1 \otimes \dots \otimes S_{n-1}}(A_1 \otimes \dots \otimes A_{n-1}, \text{Hom}_{A_n}(A_n, L_1 \otimes_{T_1} \dots \otimes_{T_{n-1}} L_n)) \cong \\ & \text{Hom}_{S_1 \otimes \dots \otimes S_{n-1}}(A_1 \otimes \dots \otimes A_{n-1}, L_1 \otimes_{T_1} \dots \otimes_{T_{n-1}} \text{Hom}_{A_n}(A_n, L_n)) \cong \\ & \text{Hom}_{S_1 \otimes \dots \otimes S_{n-1}}(A_1 \otimes \dots \otimes A_{n-1}, L_1 \otimes_{T_1} \dots \otimes_{T_{n-2}} L_{n-2}) \otimes_{T_{n-1}} \text{Hom}_{A_n}(A_n, L_n). \end{aligned}$$

By induction on n we get

$$\begin{aligned} & \text{Hom}_{S_1 \otimes \dots \otimes S_n}(A_1 \otimes \dots \otimes A_n, L_1 \otimes_{T_1} \dots \otimes_{T_{n-1}} L_n) \cong \\ & \text{Hom}_{S_1}(A_1, L_1) \otimes_{T_1} \dots \otimes_{T_{n-1}} \text{Hom}_{S_n}(A_n, L_n). \end{aligned} \tag{13}$$

Let now X_{ij} be modules as in (5). So we have rings R_{ij} , $0 \leq i \leq n$, $1 \leq j \leq m_n$ with $R_{im_i} = R_{(i+1)0} = R_i$ and X_{ij} is a module $R_{ij-1}^o \otimes R_{ij} \dashrightarrow R_{ij-1}^o \otimes R_{ij}$. In the above, set

$$\begin{aligned} T_i &= R_{i1} \otimes \dots \otimes R_{im_i-1} \\ S_i &= R_{i1}^o \otimes R_{i1} \otimes \dots \otimes R_{im_i-1}^o \otimes R_{im_i-1}, \\ A_i &= e \otimes \dots \otimes e, \\ L_i &= X_{i1} \otimes_{R_{i1}} \dots \otimes_{R_{im_i-1}} X_{im_i}. \end{aligned}$$

Then

$$\text{Hom}_{S_i}(A_i, L_i) = X_{i0} \times_{R_{i1}} \dots \times_{R_{im_i}} X_{im_i}.$$

If $X_{ij}^{R_{ij}}$ are projective then L_i is projective by Lemma 13. If $X_{ij}^{R_{ij}^o}$ are heredity and R_i^o is right heredity then $\text{Hom}_{S_i}(A_i, L_i)^{R_i^o}$ is heredity by Lemma 16 and 18. It follows that there is an isomorphism (13). The map $\beta_{m_1+\dots+m_n}$ is the result of application of

$$\text{Hom}_{(S_1 \otimes \dots \otimes S_{n-1})}(A_1 \otimes \dots \otimes A_{n-1}, -)$$

to this isomorphism and hence an isomorphism itself.

Suppose now that the X_{ij} are unions of projectives both as left and right modules and the base rings R_{ij} are heredity. Each of X_{ij} can be written as a union of submodules which are projective both as left and right modules. Then since R_{ij} and L_i are unions of projectives and hence flat both sides in (5) are unions of submodules obtained by varying the arguments in (5) to projective submodules. Restrictions of (5) to these submodules are isomorphisms hence (5) is an isomorphism itself. \square

Algebroids and quantum categories in $(k\text{-Mod})^{\text{op}}$ are the \times_A -coalgebras and \times_R -bialgebras of Takeuchi [9], these were later called coalgebroids and bialgebroids respectively.

Definition 20. A comodule between coalgebroids is a module between algebroids in $(k\text{-Mod})^{\text{op}}$.

Definition 21. A comodule algebra between bialgebroids is a quantum module between quantum categories in $(k\text{-Mod})^{\text{op}}$.

The operation \bullet is defined by the equalizer

$$M_1 \bullet M_2 \longrightarrow M_1 \times_R M_2 \rightrightarrows M_1 \times_R A \times_R N_M$$

The class of all those modules $X : R^o \otimes R \dashrightarrow R^o \otimes R$ where R is a hereditary ring and X_R and X_{R^o} are unions of projectives satisfies all the conditions of Theorem 10. From Theorem 10 and Theorem 11 we get:

Theorem 22. Coalgebroids (A, R) with R a hereditary ring and A^R and A^{R^o} unions of projectives, and comodules between bialgebroids M with M^R and M^{R^o} unions of projectives, form a bicategory.

Theorem 23. Bialgebroids (A, R) with R a hereditary ring and A^R and A^{R^o} unions of projectives, and comodule algebras between bialgebroids M with M^R and M^{R^o} unions of projectives, form a bicategory.

6 Modules of weak bialgebras

Cauchy completion \mathcal{QV} of a category \mathcal{V} is the category whose objects are pairs (X, e) , where X is an object and $e : X \longrightarrow X$ is an idempotent. A morphism $f : (X, e) \longrightarrow (Y, e')$ in \mathcal{V} is a map $f : X \longrightarrow Y$ such that $e'fe = f$. Note that the identity on (X, e) is e . Idempotents split in \mathcal{QV} .

We will assume that idempotents split in \mathcal{V} itself. In this case \mathcal{QV} is equivalent to \mathcal{V} . Using this equivalence we will sometimes identify an object (X, e) of \mathcal{QV} with its splitting in \mathcal{V} .

A parallel pair of morphisms $f_1, f_2 : X \longrightarrow Y$ in \mathcal{V} is called cosplit if there exists an arrow $d : Y \longrightarrow X$ such that

$$df_1 = 1 \quad f_1df_2 = f_2df_2$$

The map df_2 is an idempotent whose splitting provides a cosplit coequalizer for the pair f_1, f_2 .

A Frobenius monoid in a monoidal category \mathcal{V} is an object C with a monoid and a comonoid structures on it related by

$$\begin{array}{c} | \\ \diagdown \\ | \end{array} = \begin{array}{c} | \\ \diagup \\ | \end{array}$$

A Frobenius monoid is separable if additionally

$$\begin{array}{c} | \\ \diagdown \\ | \\ \diagup \\ | \end{array} = \begin{array}{c} | \end{array}$$

Every separable Frobenius monoid C is self-dual in \mathcal{V} with unit and counit:

$$\begin{array}{c} \diagup \\ \diagdown \\ | \\ \circ \end{array} \quad \begin{array}{c} \circ \\ | \\ \diagup \\ \diagdown \end{array}$$

Suppose that X is a right C -comodule and Y is a left C -comodule. If C is separable Frobenius, then the pair of morphisms

$$X \otimes Y \xrightleftharpoons[\delta_l^Y]{\delta_r^X} X \otimes C \otimes Y$$

is cosplit by $(1 \otimes \mu \otimes 1)(1 \otimes 1 \otimes \delta_l)$. The induced idempotent a on $X \otimes Y$ in the string notation is

$$\begin{array}{c} X \quad Y \\ | \quad | \\ \diagdown \quad \diagup \\ C \quad C \\ | \quad | \\ \diagup \quad \diagdown \\ C \quad C \\ | \quad | \end{array}$$

So if C is a separable Frobenius, then we have

$$X \otimes_C Y = (X \otimes Y, a). \tag{14}$$

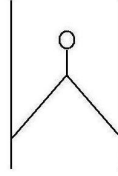
Suppose that X is a right C -module and Y is a left C -module. $X \otimes_C Y$ is defined by a coequalizer of the pair

$$X \otimes C \otimes Y \rightrightarrows X \otimes Y. \quad (15)$$

Very much like the case of comodules if C is a separable Frobenius monoid then

$$X \otimes Y = (X \otimes Y, b), \quad (16)$$

where b is the idempotent on $X \otimes Y$:

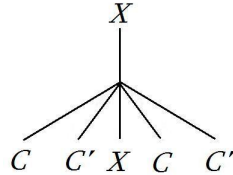


Henceforce C will be a separable Frobenius monoid.

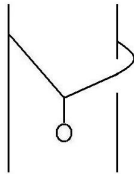
The coaction of a comodule

$$X : C^o \otimes C' \rightarrowtail C^o \otimes C'$$

in string notation is



Since such a comodule is regarded as a left C -comodule and a right C -comodule using the right $C^o \otimes C$ coaction on it, the tensor product $X_1 \otimes_C X_2$ over C will be $(X_1 \otimes X_2, a)$, with a being the idempotent

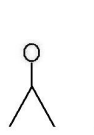


Note that here we have taken a free hand with string notation. In the above diagram it is not clear that on the left string we are using the right C -coaction and on the right string we are using the right C^o -coaction. However this should be clear from the context. The same occurs below.

Since C is selfdual in \mathcal{V} , $\text{coHom}(C, X)$ exists for every X and is given by

$$\text{coHom}(C, X) = C \otimes X,$$

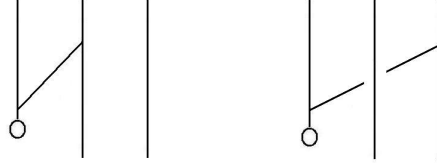
with coevaluation



The diagram (7) becomes

$$(C \otimes X_1 \otimes X_2, C \otimes a) \xrightleftharpoons[(\delta_l^{C^o})^*]{(\delta_l^C)^*} (X_1 \otimes X_2, a). \quad (17)$$

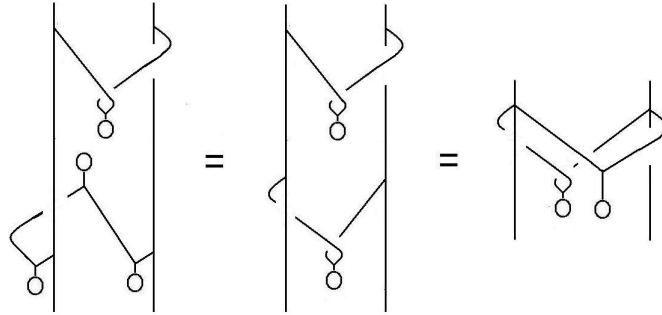
The maps $(\delta_l^C)^*, (\delta_l^{C^o})^* : C \otimes X_1 \otimes X_2 \longrightarrow X_1 \otimes X_2$ are



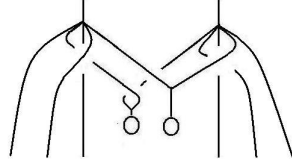
Up to precomposing with the isomorphism $c \otimes X_2$ the pair $(\delta_l^C)^*, (\delta_l^{C^o})^*$ is an instance of the pair (15) in \mathcal{V} . So the coequalizer is computed by (16). Taking the coequalizer of (17) we get

$$T_2(X_1, X_2) = (X_1 \otimes X_2, d_2),$$

where the idempotent d_2 is



$T_2(X_1, X_2)$ is a comodule $C_0^o \otimes C_2' \twoheadrightarrow C_0^o \otimes C_2'$ with coaction $(X_1 \otimes X_2, d) \twoheadrightarrow (C_0 \otimes X_1 \otimes X_2 \otimes C_2, 1 \otimes d_2 \otimes 1)$:

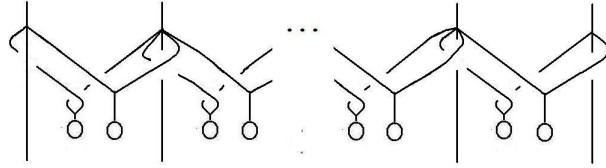


Generally we have:

Theorem 24. *If the base comonoids are separable Frobenius monoids, then for $n \geq 2$*

$$T_n(X_1 \dots X_n) = (X_1 \otimes \dots \otimes X_n, d_n),$$

where d_n is the idempotent



We can see that if the base comonoids are Frobenius separable, then for a partition ξ not involving parts of zero length the map β_ξ (5) is an isomorphism.

$T_C() = C \otimes C$ with left and right $C^o \otimes C$ -coactions:



The next proposition asserts that the class of all comodules $C^o \otimes C' \twoheadrightarrow C^o \otimes C'$ with C a Frobenius monoid satisfies condition 3 in Theorem 10. The remaining conditions are obviously satisfied.

Proposition 25. *Suppose that X is a right C -comodule. If C is a separable Frobenius monoid and $X \otimes - : \mathcal{V} \twoheadrightarrow \mathcal{V}$ preserves reflexive coequalizers, then the functor*

$$X \otimes_C - : \text{Comod}\mathcal{V}(C, I) \twoheadrightarrow \mathcal{V}$$

preserves reflexive coequalizers.

Proof. Let

$$X_1 \rightrightarrows X_2 \longrightarrow X_3$$

be a reflexive coequalizer in $\text{Comod}\mathcal{V}(C, I)$. It is computed as a coequalizer in \mathcal{V} . Since $X \otimes -$ preserves reflexive coequalizers

$$X \otimes X_1 \rightrightarrows X \otimes X_2 \longrightarrow X \otimes X_3$$

also is a coequalizer. Further, we have a coequalizer in \mathcal{QV} :

$$(X \otimes X_1, a_1) \rightrightarrows (X \otimes X_2, a_2) \longrightarrow (X \otimes X_3, a_3)$$

where a_i are idempotents as in (14). By splitting these idempotents we prove that

$$X \otimes_C X_1 \rightrightarrows X \otimes_C X_2 \longrightarrow X \otimes_C X_3$$

is a coequalizer in \mathcal{V} . □

In [5] it was shown that a quantum category with a separable Frobenius base monoid is the same as a weak bialgebra in \mathcal{V} , which is an object A with a comonoid and monoid structures on it related in a certain way.

Definition 26. A module between weak bialgebras is a module between quantum categories with a separable Frobenius base monoid.

Definition 27. A module comonoid between weak bialgebras is a quantum module between quantum categories with separable Frobenius base comonoid.

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